2. MathPages.com. "The Dullness of 1729." *http://www.mathpages.com/home/kmath028.htm*.

*—Moses Klein and Mark Purtill*

## HACK #37 Test for Divisibility

It's often useful to know whether one number is evenly divisible by another number. Here are some tricks that go beyond knowing whether a number is odd or even, or divisible by 10.

Before decimals such as 3.5 were invented, people had to use numbers with fractional parts instead, such as $3\frac{1}{2}$. In many division problems, they had to reduce fractions with large numbers—for example, 243 / 405—to their lowest terms—in this case, $\frac{3}{5}$. Knowing rules to determine divisibility by the integers from 1 through 12, or from 1 through 15, was very useful in that precalculator time.[1]

If you want to strengthen your mental math powers, knowing the same rules can be useful to you today. In particular, these rules are helpful with math tricks that involve factoring numbers, such as simplified mental multiplication. Sometimes, knowing *that* a number is evenly divisible by another number goes at least halfway toward knowing *what* the answer is.

### In Action

The following list gives tests for divisibility by all integers from 1 to 15. In this context, *divisible* means *evenly divisible*—that is, divisible with a remainder of 0.

1. Every integer is divisible by 1.
2. If the number's last digit is even (0, 2, 4, 6, or 8), the number is divisible by 2. *Examples:* 22, 136, 54, 778.
3. If the number's digit sum is 0, 3, or 6 (or 9, which is the same as 0 for this purpose), the number is divisible by 3. (See "Calculate Mental Checksums" **[Hack #38]** for how to calculate digit sums.) *Example:* 138 ($1 + 3 + 8 = 12$; $1 + 2 = 3$).
4. If the last two digits of the number, taken as a two-digit number, are divisible by 4, so is the number. *Example:* 216 (16 is divisible by 4).
5. If the last digit of a number is 0 or 5, the number is divisible by 5. *Example:* 147,325 (the last digit is 5).
6. If a number is divisible by both 2 and 3, the number is also divisible by 6. (See the tests for 2 and 3.) *Example:* 138 is divisible by 2 because its

last digit is 8. It is also divisible by 3 because 1 + 3 + 8 = 12 and 1 + 2 = 3. Therefore, it's also divisible by 6.

7. See the "Divisibility by 7" sidebar.
8. If the last three digits of the number, taken as a three-digit number, are divisible by 8, so is the number. *Example:* 2,889,888 (the last three digits, 888, are divisible by 8).
9. If the number's digit sum is 0 (or 9, which is the same as 0 for this purpose), the number is divisible by 9. (See "Calculate Mental Checksums" [Hack #38] for how to calculate digit sums.) *Example:* 41,805 (4 + 1 + 8 + 0 + 5 = 18; 1 + 8 = 9).
10. If the last digit of a number is 0, the number is divisible by 10. *Example:* 99,310 (the last digit is 0).
11. Casting out elevens [Hack #38] is the easiest way to test for divisibility by 11 in most cases: if the number modulo 11 is 0, the number is divisible by 11.
12. If a number is divisible by both 3 and 4, the number is also divisible by 12. (See the tests for 3 and 4.) *Example:* 624 is divisible by 3 because 6 + 2 + 4 = 12 and 1 + 2 = 3. It is also divisible by 4 because the last two digits (24) are divisible by 4. It is therefore divisible by 12.
13. If 9 times the last digit of the number, subtracted from the number with its last digit deleted, is divisible by 13, so is the number.[2] *Example:* 351 is divisible by 13 because 35 – 9 × 1 = 26. Since 26 is divisible by 13, so is 351.
14. If a number is divisible by both 2 and 7, the number is also divisible by 14. (See the tests for 2 and 7.) *Example:* 65,282,490 is divisible by 2 because it ends in 0. It is also divisible by 7 because it is 7 less than 65,282,497, which we know is divisible by 7 from the example in the "Divisibility by 7" sidebar. Since it is divisible by both 2 and 7, it is divisible by 14.
15. If a number is divisible by both 3 and 5, the number is also divisible by 15. (See the tests for 3 and 5.) *Example:* 3,285 is divisible by 3 because 3 + 2 + 8 + 5 = 18 and 1 + 8 = 9. It is also divisible by 5 because it ends in 5. Therefore, it is divisible by 15.

## In Real Life

Here's an example of the kind of situation where knowing tests for divisibility will come in handy in real life.

## Divisibility by 7

The following procedure is one of the simplest available for testing divisibility by 7:

1. Take the number you want to test, such as *65,282,497*.
2. Split the number into groups of three digits, starting at the right. If the number is written with commas, the splits will be easy to see. Don't worry if the leftmost group has less than three digits. In this case, the groups are *497*, *282*, and *65*.
3. Alternately, add and subtract these groups, treated as three-digit numbers. To continue the example, *+497 – 282 + 65 = 280*.
4. The output of this procedure will have the same divisibility by 7 that the original number does. It's easy to see that *280 / 7 = 40*, so 280 is divisible by 7, and it follows that 65,282,497 is, too.

The same procedure will work for figuring divisibility by 11 and 13: simply test the output of this procedure for divisibility by 11 or 13, respectively, rather than 7.

You're at a dinner for 11 people. The restaurant is closing, and everyone agrees to split the bill evenly to save time, but no one has a pocket calculator or PDA handy.

The bill is $419.15, including gratuity. You round this to $419, and cast out elevens. The result is 1, which means that by subtracting 1 from 419, you'll reach a number evenly divisible by 11, which is 418. Quick mental division shows you that everyone owes $38 (418 / 11 = 38), and that if random people around the table contribute some pocket change to make up the difference of $1.15, you can pack up and get out of the restaurant quickly.

## End Notes


1. Gardner, Martin. 1991. "Tests of Divisibility." *The Unexpected Hanging and Other Mathematical Diversions*. The University of Chicago Press. An excellent article on divisibility, and a primary source for this hack. Gives the rules for 1 through 12, several additional tests for divisibility by 7, magic tricks involving divisibility, and more, in the wonderful Gardner style.
2. Wikipedia article. "Divisor." *http://en.wikipedia.org/wiki/Divisor*. Gives the rules for 13–15, defines some terminology, outlines some basic principles, and specifies a somewhat elaborate rule for determining divisibility of any integer, in any base, by any smaller integer.

HACK #38

# Calculate Mental Checksums

Computers use checksums to ensure that data was not corrupted in transmission. Now your brain can use a checksum for your mental math, with a few easy techniques.

It's important to have some way to check your mental math that doesn't take as long as solving the problem did originally, and ideally is much shorter. It's easy to check your math for the four basic operations of arithmetic (addition, subtraction, multiplication, and division) by calculating *digit sums* for the numbers involved. A digit sum is a special kind of *checksum* or data integrity check. Checksums are used all over the world of computing, from credit cards to ISBNs on books, to downloads you make with your web browser. Now your brain can use them, too.

Finding the digit sum of a number is easy. Just add all the digits of the number together. If the result is greater than 9, add the digits together again. Continue to do so until you have a one-digit result. If the result is 9, reduce it to 0. The result is the digit sum of the original number.[1]

For example, the digit sum of 381 is 3:

```
3 + 8 + 1 = 12
1 + 2 = 3
```

Similarly, the digit sum of 495 is 0:

```
4 + 9 + 5 = 18
1 + 8 = 9 (same as 0)
```

A number's digit sum is actually that number *modulo 9*—in other words, the remainder when that number is divided by 9. See "Calculate Any Weekday" [Hack #43] for a refresher on modulo arithmetic.

This technique is also known as *casting out nines*. Casting out nines and a similar technique known as *casting out elevens* (discussed in the following section) are all you need to check your arithmetic calculations rapidly and to a high degree of accuracy.

## In Action

This section shows how to calculate checksums for the four basic operations: addition, subtraction, multiplication, and division. Only integers are used in the examples, but the techniques will work just as well for real numbers as long as they have the same number of decimal places. For example, if you are multiplying 13.52 by 14.6, think of the latter number as 14.60.

**Addition.** To check your answer after addition:

1. Find the digit sums for the numbers you are adding.
2. Add all the digit sums together.
3. Find the digit sum of the new number, and the digit sum of the answer number.
4. Compare these two digit sums. If the digit sums match, the answer should be correct.

Here is an example of checking addition successfully:

```
  95  9 + 5 = 14; 1 + 4 = 5
+ 42  4 + 2 = 6
+ 22  2 + 2 = 4 ... 5 + 6 + 4 = 15; 1 + 5 = 6
 159  1 + 5 + 9 = 15; 1 + 5 = 6 : OK
```

Here is another example of checking addition:

```
  49  4 + 9 = 13; 1 + 3 = 4
+ 37  3 + 7 = 10; 1 + 0 = 1 ... 4 + 1 = 5
  76  7 + 6 = 13; 1 + 3 = 4 : WRONG (the answer should be 86)
```

The fact that the digit sum of the numbers being added does not match the digit sum of the result tells you that the result is incorrect.

Digit sums are even easier to calculate if you treat the 9s as 0s right away. Thus, instead of 4 + 9 = 13 in the example, just compute 4 + 0 = 4 and obtain your digit sum in one step.

**Multiplication.** To check your answer during multiplication:

1. Find the digit sums for the numbers you are multiplying.
2. Multiply them.
3. Find the digit sum of the new number, and the digit sum of the answer number.
4. Compare these two digit sums. If the digit sums match, the answer should be correct.

Here is an example of a correct multiplication:

```
  33  3 + 3 = 6
x 27  2 + 7 = 9 = 0 ... 6 x 0 = 0
 891  8 + 9 + 1 = 18; 1 + 8 = 9 = 0 : OK
```

Here is another example of checking multiplication:

```
  76  7 + 6 = 13; 1 + 3 = 4
x 14  1 + 4 = 5 ... 4 x 5 = 20; 2 + 0 = 2
1164  1 + 1 + 6 + 4 = 12; 1 + 2 = 3 : WRONG (the answer should be 1064)
```

The fact that the digit sum of the product of the digit sums of the numbers being multiplied does not match the digit sum of the result tells you that the result is incorrect.

**Subtraction.** Subtraction is the inverse of addition. Checking a subtraction problem works the same way as checking an addition problem: simply turn the subtraction problem into an addition problem first, as in the following example:

```
  58         26  2 + 6 = 8
- 26   →   + 32  3 + 2 = 5 ... 8 + 5 = 13; 1 + 3 = 4
  32         58  5 + 8 = 13;  1 + 3 = 4 : OK
```

**Division.** Just as subtraction is the inverse of addition, so is division the inverse of multiplication. Thus, checking a division problem works the same way as checking a multiplication problem: first turn the division problem into a multiplication problem, as in the following example:

```
    1081 / 23 = 46

      23  2 + 3 = 5
    x 46  4 + 6 = 10;   1 + 0 = 1 ... 5 x 1 = 5
    1081  1 + 0 + 8 + 1 = 10; 1 + 0 = 1 : WRONG
(the answer should be 47, not 46)
```

The fact that the digit sum of the product of the digit sums of the numbers being multiplied does not match the digit sum of the result tells you that the division result of 46 is incorrect.

**False positives.** Sometimes casting out nines will not find an error. For example, sometimes errors in two digits will cancel out, as in the following example:

```
  272  2 + 7 + 2 = 11; 1 + 1 = 2
+ 365  3 + 6 + 5 = 14; 1 + 4 = 5 ... 2 + 5 = 7
  547  5 + 4 + 7 = 16; 1 + 6 = 7 : OK?
```

The correct answer is not 547, but 637 (6 + 3 + 7 = 16; 1 + 6 = 7).

Casting out nines will also not find *errors of place* (when the decimal point is misplaced, or the result is otherwise off by a power of 10). Estimating the order of magnitude [Hack #41] (roughly, finding the number of digits to the left of the decimal point) will help catch some errors of place, but casting out elevens is even better.

**Casting out elevens.** Casting out elevens (that is, calculating numbers *modulo 11*) is slightly more accurate than casting out nines. It will also catch errors that casting out nines will not, including errors of place, so it is useful as a cross-check.

To cast out elevens from an integer, simply add all of the digits in the odd places of the number (for example, the ones and hundreds digits, which are in places 1 and 3, counting from the right), then subtract all of the digits in the even places (for example, the tens and thousands digits, in places 2 and 4).

For example, casting out elevens from 5,924 gives the result 4 + 9 – 2 – 5 = 6.

If your result is greater than 11, cast out elevens from *that* number, and continue doing so until you have a number that's less than 11. If the result is less than 0, add 11 to it until you have a number that is at least 0 and less than 11. To cast out elevens from a sum, total the result of casting out elevens for each number you're adding, and then cast out elevens from that.

The following example is a cross-check of the sum from the previous section:

```
  272  2 + 2 - 7 = -3; -3 + 11 = 8
+ 365  5 + 3 - 6 = 2 ... 8 + 2 = 10
  547  7 + 5 - 4 = 8 : WRONG (the answer should be 637)
```

## How It Works

A rigorous mathematical proof that casting out nines by summing the digits of a number will give you that number's remainder when divided by 9 is beyond the scope of this book, but it's fairly intuitive.

First, consider that `0 mod 9` is `0`, `1 mod 9` is `1`, `2 mod 9` is `2`, `3 mod 9` is `3`, and so on. `9 mod 9` is `0`, `10 mod 9` is `1`, `11 mod 9` is `2`, and the cycle continues.

Next, consider that `20 mod 9` is `2`, `30 mod 9` is `3`, and so on; check and see. You will also find that `200 mod 9` is `2`, `2,000 mod 9` is `2`, `20,000 mod 9` is `2`, and so on. In fact, any integer multiplied by any power of 10 and then calculated modulo 9 has the same result as the original integer modulo 9.

Since `(a + b + c) mod 9` is the same as `(a mod 9) + (b mod 9) + (c mod 9)`, and since (for example) the number 523 can be written as `500 + 20 + 3`, simply adding the individual digits of 523 `(5 + 2 + 3)` will serve the same purpose as calculating the sum of 500 mod 9, 20 mod 9, and 3 mod 9, to wit, finding 523 modulo 9.[2]

Figuring out why casting out elevens works is left as an exercise for the mathematically minded mind-performance hacker.

## In Real Life

Mental checksums are most useful when you combine them with other math hacks [Hack #75]. Since checksums using both 9 and 11 are easy to find, checking your work won't add much time to your mental calculations unless you made a mistake—in which case, better late than wrong.

## End Notes


1. Julius, Edward H. 1992. *Rapid Math Tricks and Tips*. John Wiley & Sons, Inc.
2. Menninger, Karl, and E. J. F. Primrose (trans). 1961. *Calculator's Cunning: The Art of Quick Reckoning*. Basic Books, Inc., Publishers.


HACK #39

# Turn Your Hands into an Abacus

You might have heard stories of how rapid and accurate calculations with an abacus can be, but did you know that the abacus might have been based on an ancient technique using only the human hand, which survives today as the Korean art of Chisenbop?

*Chisenbop* is an ancient Korean technique for calculations with the human hand. The classic text on Chisenbop in English is *The Complete Book of Fingermath*.[1] Unfortunately, it is expensive, aimed toward children, and takes hundreds of pages to explain principles that an educated adult can learn in a few minutes. One important thing that the book can offer you, however, is page after page of drills. Chisenbop should become a motor skill, not something you have to think about.

*Fingermath* also uses many full, detailed drawings of hands in action, which is another reason the book is so long. Fortunately, the Wikipedia presents a notation that can radically compress Chisenbop diagrams on the page, as shown in Table 4-2.[2]

I added a couple of symbols to the notation myself for this book (^ and v, which are described in Table 4-2).

*Table 4-2 . Chisenbop notation*

| Notation | Meaning |
|---|---|
| - | A thumb in the air |
| @ | A thumb with its tip pressed to the table |
| . | A finger in the air |
| o | A finger with its tip pressed to the table |
| ^ | Lift that finger |
| v | Press that finger down |

You can combine the finger notation across two hands, as shown in the examples in Table 4-3.